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Spaces of compact operators on $C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha])$ spaces

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ABSTRACT

We classify up to isomorphism the spaces of compact operators $\mathcal{K}(E,F)$, where E and F are Banach spaces of all continuous functions defined on the compact spaces $\mathbf{2}^{\mathfrak{m}} \oplus [0,\alpha]$, the topological sum of Cantor cubes $\mathbf{2}^{\mathfrak{m}}$ and the intervals of ordinal numbers $[0,\alpha]$. More precisely, we prove that if $2^{\mathfrak{m}}$ and $\aleph_\gamma$ are not real-valued measurable cardinals and $\mathfrak{n} \geqslant \aleph_0$ is not sequential cardinal, then for every ordinals ξ, η, λ and μ with $\xi \geqslant \omega_1$, $\eta \geqslant \omega_1$, $\lambda = \mu < \omega$ or $\lambda, \mu \in [\omega_\gamma, \omega_{\gamma+1}[$, the following statements are equivalent:

(a) $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]))$ and $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\mu]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\eta]))$ are isomorphic.
(b) Either $C([0,\xi])$ is isomorphic to $C([0,\eta])$ or $C([0,\xi])$ is isomorphic to $C([0,\alpha p])$ and $C([0,\eta])$ is isomorphic to $C([0,\alpha q])$ for some regular cardinal α and finite ordinals $p \neq q$.

Thus, it is relatively consistent with ZFC that this result furnishes a complete isomorphic classification of these spaces of compact operators.

1. Introduction

In this paper we use standard notation in set theory and Banach spaces theory as in [14] and [15] respectively. Let X be a Banach space and K a compact Hausdorff space. By $C(K,X)$ we denote the Banach space of all continuous X-valued functions defined on K and equipped with the supremum norm. This space will be denoted by $C(K)$ in the case $X = \mathbb{R}$. Given Banach spaces X and Y, $\mathcal{K}(X,Y)$ denotes the Banach space of compact operators from X to Y. For a fixed cardinal $\mathfrak{m}$, $\mathbf{2}^{\mathfrak{m}}$ denotes the product of $\mathfrak{m}$ family of copies of the two-point spaces $\mathbf{2}$, endowed with the product topology. Let α be an ordinal number, by $[0,\alpha]$ we denote the interval of ordinals $\{\xi\colon 0 \leqslant \xi \leqslant \alpha\}$ endowed with the order topology. As usual, if K_1 and K_2 are compact spaces, we denote by $K_1 \oplus K_2$ the topological sum of these spaces. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic. Finally, we recall that cardinals simply are initial ordinals. However, in this paper, it is ordinal not cardinal arithmetic that is in use.

The present paper is a continuation of [10] where it has been shown that under a certain condition on the cardinal $\mathfrak{n}$, for every uncountable ordinals ξ and η we have the following cancellation law:

$$C\big(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]\big) \sim C\big(\mathbf{2}^{\mathfrak{n}} \oplus [0,\eta]\big) \quad\Longleftrightarrow\quad C\big([0,\xi]\big) \sim C\big([0,\eta]\big). \tag{1}$$

Therefore the isomorphic classification of the $C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\alpha])$ spaces, with $\alpha \geqslant \omega_1$, is reduced to the isomorphic classification of the $C([0,\alpha])$ spaces. The isomorphic classification of these last spaces was accomplished by Bessaga and Pełczyński [2], in the case where $\omega \leqslant \alpha < \omega_1$; Semadeni [23], in the case where $\omega_1 < \alpha \leqslant \omega_1 \cdot \omega$; Labbé [18], in the case where $\omega_1 \cdot \omega < \alpha < \omega_1^{\omega}$ and independently Kislyakov [17] and Gul'ko and Os'kin [13], in the general case.

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The main purpose of this paper is to extend the cancellation law (1) involving $C(K)$ spaces to a cancellation law involving $\mathcal{K}(X,Y)$ spaces. As an application of this new cancellation law, we show that under some conditions the isomorphic classification of the spaces of compact operators between $C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\alpha])$ spaces, with $\alpha \geqslant \omega_1$, is also reduced to the isomorphic classification of the $C([0,\alpha])$ spaces.

In order to this we will prove the following theorem which in the particular case $Y = \mathbb{R}$ is [10, Theorem 1.6].

Theorem 1.1. *Let X and Y be Banach spaces having the Mazur property such that Y contains no subspace isomorphic to c_0 and ξ and η uncountable ordinals. Then the following are equivalent*:

(a) $X \oplus C([0,\xi],Y) \sim X \oplus C([0,\eta],Y)$.
(b) *Either $C([0,\xi]) \sim C([0,\eta])$ or $C([0,\xi]) \sim C([0,\alpha p])$, $C([0,\eta]) \sim C([0,\alpha q])$ and $Y^p \sim Y^q$ for some regular cardinal α and finite ordinals $p \neq q$.*

We recall that a Banach space X is said to have the Mazur property (in short MP), if every element of X^{**}, the bidual space of X, which is sequentially weak* continuous is weak* continuous and thus is an element of X. Such spaces were investigated in [7,19] and also in [16] and [26] where they were called d-complete and μB-spaces, respectively.

Remark 1.2. Notice that the two conditions of statement (b) of Theorem 1.1 are mutually exclusive. Indeed, if p and q are two different finite ordinals and α is an uncountable regular cardinal, then by [17, Theorem 2] $C([0,\alpha p]) \nsim C([0,\alpha q])$.

Remark 1.3. Observe that the second condition of the statement (b) of Theorem 1.1 cannot be removed even in the case where $X = \{0\}$. Indeed, if Y is a Banach space isomorphic to its square Y^2, then $C([0,\omega_1],Y) \sim C([0,\omega_1], Y \oplus Y)$ and therefore

$$C\big([0,\omega_1],Y\big) \sim C\big([0,\omega_1],Y\big) \oplus C\big([0,\omega_1],Y\big) \sim C\big([0,\omega_1] \oplus [0,\omega_1],Y\big) \sim C\big([0,\omega_1 \cdot 2],Y\big).$$

But according to [17, Theorem 2], $C([0,\omega_1]) \nsim C([0,\omega_1 \cdot 2])$.

To state the above mentioned cancellation law we need to recall that following Noble [25] and Antonowskij–Chudnowsky [1], we say that a cardinal $\mathfrak{m}$ is sequential if there exists a sequentially continuous but not continuous real-valued function on $\mathbf{2}^{\mathfrak{m}}$. Recall that a function $f : \mathbf{2}^{\mathfrak{m}} \to \mathbb{R}$ is said to be sequentially continuous when $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to k in $\mathbf{2}^{\mathfrak{m}}$. The first sequential cardinal will be denoted by $\mathfrak{s}$.

We also need to recall that an uncountable cardinal $\mathfrak{m}$ is real-valued measurable if there exists a nontrivial $\mathfrak{m}$-additive measure μ on $\mathfrak{m}$ [14, page 300]. We will denote by $\mathfrak{m}_{\mathbb{R}}$ the least real-valued measurable cardinal.

The cancellation law (1) follows immediately from the following corollary, see the proof of the case where $\mathfrak{m}$ and λ are finite.

Corollary 1.4. *Suppose that $2^{\mathfrak{m}} < \mathfrak{m}_{\mathbb{R}}$, $\aleph_\gamma < \mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{n} < \mathfrak{s}$. Then for every ordinals ξ, η, λ and μ with $\xi \geqslant \omega_1$, $\eta \geqslant \omega_1$, $\lambda = \mu < \omega$ or $\lambda, \mu \in [\omega_\gamma, \omega_{\gamma+1}[$, the following assertions are equivalent*:

(a) $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi])) \sim \mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\mu]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\eta]))$.
(b) *Either $C([0,\xi]) \sim C([0,\eta])$ or $C([0,\xi]) \sim C([0,\alpha p])$ and $C([0,\eta]) \sim C([0,\alpha q])$ for some regular cardinal α and finite ordinals $p \neq q$.*

Proof. It will be convenient to consider three cases:

Case 1. $\mathfrak{m}$ is infinite. By [22, Proposition 5.2] we know that

$$\big(C\big(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]\big)\big)^* \sim \bigg(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\bigg)_1 \oplus l_1[0,\lambda].$$

In addition, [6, Example 11, page 245] guarantees that $L_1[0,1]^{\mathfrak{m}}$ and $l_1[0,\lambda]$ have the approximation property. Therefore according to [3, Proposition 2.14], the spaces $(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]))^*$ have also the approximation property. Then we can apply [5, Proposition 5.3] to identify the spaces of compact operators which are considering as an injective tensor product of Banach spaces as follows

$$\mathcal{K}\big(C\big(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]\big), C\big(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]\big)\big) \sim \bigg(\bigg(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\bigg)_1 \oplus l_1[0,\lambda]\bigg) \hat{\hat{\otimes}} \big(C\big(\mathbf{2}^{\mathfrak{n}}\big) \oplus C\big([0,\xi]\big)\big).$$

If we denote

$$X = \bigg(\bigg(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\bigg)_1 \oplus l_1[0,\lambda]\bigg) \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \sim \bigg(\bigg(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\bigg)_1 \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big)\bigg) \oplus \big(l_1[0,\lambda] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big)\big),$$

and

$$Y = \left(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\right)_1 \oplus l_1[0,\lambda],$$

then by [6, Example 6, page 224] we can write

$$\mathcal{K}\big(C\big(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]\big), C\big(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]\big)\big) \sim X \oplus C\big([0,\xi], Y\big). \tag{2}$$

But by our hypotheses on λ and μ it follows that

$$l_1[0,\lambda] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \sim l_1[0,\mu] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big),$$

and

$$\left(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\right)_1 \oplus l_1[0,\lambda] \sim \left(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\right)_1 \oplus l_1[0,\mu].$$

Thus analogously to (2) we have

$$\mathcal{K}\big(C\big(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]\big), C\big(\mathbf{2}^{\mathfrak{n}} \oplus [0,\mu]\big)\big) \sim X \oplus C\big([0,\eta], Y\big). \tag{3}$$

Moreover, by [16, Corollary 5.2] $L_1[0,1]^{\mathfrak{m}}$ has MP and since $|\lambda| < \mathfrak{m}_{\mathbb{R}}$, $l_1[0,\lambda]$ has MP [7, Theorem 5.10]. Since $2^{\mathfrak{m}} < \mathfrak{m}_{\mathbb{R}}$, [16, Theorem 3.1] ensures that Y has MP.

Next we will establish that X also has MP. Since $\mathfrak{n} < \mathfrak{s}$, we have that $C(\mathbf{2}^{\mathfrak{n}})$ has MP [20], see also [21, Proposition 5.2.c]. Another appeal to [16, Corollary 5.2] shows that $L_1[0,1]^{\mathfrak{m}} \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}})$ has MP. So by [16, Theorem 5.3] $(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}})_1 \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}})$ has MP. On the other hand, again by [16, Theorem 5.3] we infer that $l_1[0,\lambda] \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}})$ has MP. Consequently X has MP as advertised.

Now observe that Y contains no copy of c_0 and $Y^p \sim Y^q$ for every finite ordinals p and q. So by (2), (3) and Theorem 1.1 we conclude that the assertions (a) and (b) are equivalent.

Case 2. $\mathfrak{m}$ is finite and λ is infinite. In this case, it suffices to take

$$X = \mathbb{R}^{2^{\mathfrak{m}}} \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \oplus l_1[0,\lambda] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \sim l_1[0,\lambda] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \sim l_1[0,\mu] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big),$$

and

$$Y = \mathbb{R}^{2^{\mathfrak{m}}} \oplus l_1[0,\lambda] \sim l_1[0,\lambda] \sim l_1[0,\mu].$$

Then again (2) and (3) hold and proceeding as in Case 1 we are done.

Case 3. $\mathfrak{m}$ and λ are finite. Thus take

$$X = \mathbb{R}^{2^{\mathfrak{m}}+\lambda+1} \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \oplus l_1[0,\lambda] \hat{\hat{\otimes}} C\big(\mathbf{2}^{\mathfrak{n}}\big) \sim C\big(\mathbf{2}^{\mathfrak{n}}\big),$$

and

$$Y = \mathbb{R}^{2^{\mathfrak{m}}+\lambda+1} \sim \mathbb{R}^{2^{\mathfrak{m}}+\mu+1}.$$

So once again (2) and (3) hold. Further, in this case, observe that $Y^p \nsim Y^q$ for every finite ordinals $p \neq q$. Therefore by Theorem 1.1 we see that following are equivalent:

(a) $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi])) \sim \mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\mu]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\eta]))$.
(b) $C([0,\xi]) \sim C([0,\eta])$. $\square$

Remark 1.5. Mazur [24] showed that $\mathfrak{s}$ is weakly inaccessible. Hence $\omega_1 < \mathfrak{s}$. Moreover, there are many weakly inaccessible cardinals before $\mathfrak{s}$ [4]. On the other hand, recall that a cardinal $\mathfrak{m}$ is two-valued measurable cardinal if there is a nontrivial two-valued $\mathfrak{m}$-additive measure defined on all subsets of a set of cardinal $\mathfrak{m}$ for which points have measure zero [7, page 560]. Let $\mathfrak{m}_2$ denote the least two-valued measurable cardinal. It is well known that $\mathfrak{s} \leqslant \mathfrak{m}_{\mathbb{R}}$; $\mathfrak{s} \leqslant 2^{\aleph_0}$ or $\mathfrak{s} = \mathfrak{m}_2$; and $\mathfrak{s} = \mathfrak{m}_2$ under Martin's axiom [1,8,24]. Thus, it is relatively consistent with ZFC (Zermelo–Fraenkel set theory plus the axiom of choice) that there exists no sequential cardinals [21] and therefore no real-valued measurable cardinal too. Hence it is also consistent with ZFC that Corollary 1.4 furnishes a complete isomorphic classification of the spaces $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]))$, with $\lambda < \omega$ or $\lambda \in [\omega_\gamma, \omega_{\gamma+1}[$, for every $\gamma \geqslant 1$ and $\xi \geqslant \omega_1$.

In view of Remark 1.5, the following question arises naturally:

Problem 1.6. Does the above isomorphic classification of the spaces of compact operators $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \oplus [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \oplus [0,\xi]))$ remain true without the hypotheses on the cardinal numbers?

2. Some preliminary results

From now on, following Bessaga and Pełczyński [2], it will be useful to denote the space $C([0,\alpha], X)$ by X^{α}.

In order to prove Theorem 1.1 we will state several lemmas. Let X and Y be Banach spaces. By $X \hookrightarrow Y$ we mean that Y has a subspace isomorphic to X. The first lemma is inspired by [2, Lemma 2] and [10, Lemma 2.1].

Lemma 2.1. *Let X and Y be Banach spaces and $\lambda \geqslant \omega$. Then*

$$\mathbb{R}^{\lambda^{\omega}} \hookrightarrow X \oplus Y^{\lambda} \quad \textit{implies that} \quad \mathbb{R}^{\lambda} \hookrightarrow X \oplus Y^{\gamma} \quad \textit{for some } \gamma < \lambda.$$

Proof. For every $\xi < \lambda$, denote $Y_0^{\xi} = \{f \in Y^{\xi}\colon\ f(\xi) = 0\}$. By [2, Lemma 1.2], Y_0^{ξ} is isomorphic to Y^{ξ}. So, proceeding by contradiction, suppose that

$$\mathbb{R}^{\lambda} \not\hookrightarrow X \oplus Y_0^{\xi}, \quad \forall \xi,\ \xi < \lambda. \tag{4}$$

According to our hypothesis there are operators $\pi_1 : \mathbb{R}^{\lambda^{\omega}} \to X$, $\pi_2 : \mathbb{R}^{\lambda^{\omega}} \to Y_0^{\lambda}$ and $a \in \mathbb{R}_+$ such that for every $f \in \mathbb{R}^{\lambda^{\omega}}$,

$$a\|f\| \leqslant \max\bigl\{\bigl\|\pi_1(f)\bigr\|, \bigl\|\pi_2(f)\bigr\|\bigr\} \leqslant \|f\|. \tag{5}$$

Fix an integer N and $\epsilon > 0$ such that $1 + \epsilon < aN$. For every $0 \leqslant \xi < \lambda$, write

$$\Delta_{\xi}^{1} = \bigl(\lambda^{N}\xi, \lambda^{N}(\xi + 1)\bigr].$$

Let Z_N be given by

$$\bigl\{f \in \mathbb{R}^{\lambda^{\omega}}\colon\ \forall \xi \in [0,\lambda)\ f \text{ is constant on } \Delta_{\xi}^{1} \text{ and } f(\xi) = 0,\ \forall \xi \in \bigl[\lambda^{N+1}, \lambda^{\omega}\bigr]\bigr\}.$$

Clearly Z_N is isomorphic to $\mathbb{R}^{\lambda}$. Thus by (4) π_1 restricted to Z_N is not an isomorphism of Z_N into X. So there exists $f_1 \in Z_N$ such that $\|f_1\| = 1$ and $\|\pi_1(f_1)\| \leqslant \epsilon/2$.

We may change f_1 to $-f_1$ and assume that there exists $\xi_1 \in [0,\lambda)$ such that for every $\gamma \in (\lambda^{N}\xi_1, \lambda^{N}(\xi_1 + 1)]$, we have $f_1(\gamma) = 1$.

Since $\pi_2(f_1) \in Y_0^{\lambda}$, there exists $\lambda_1 < \lambda$ such that for every $\gamma \in [\lambda_1 + 1, \lambda]$, we have $\|\pi_2(f_1)(\gamma)\| \leqslant \epsilon/2$.

For the second step, for every $0 \leqslant \xi < \lambda$, write

$$\Delta_{\xi}^{2} = \bigl(\lambda^{N}\xi_1 + \lambda^{N-1}\xi, \lambda^{N}\xi_1 + \lambda^{N-1}(\xi + 1)\bigr].$$

Let Z_{N-1} be defined by

$$\bigl\{f \in \mathbb{R}^{\lambda^{\omega}}\colon\ \forall \xi \in [0,\lambda)\ f \text{ is constant on } \Delta_{\xi}^{2} \text{ and } f(\xi) = 0,\ \forall \xi \notin (\lambda^{N}\xi_1, \lambda^{N}(\xi_1 + 1)]\bigr\}.$$

Denote by P_{λ_1} the natural projection of Y_0^{λ} onto $Y_0^{\gamma_1}$ and define the operator $\pi_1 + P_{\gamma_1}\pi_2 : \mathbb{R}^{\lambda^{\omega}} \to X \oplus Y_0^{\lambda_1}$ by

$$(\pi_1 + P_{\gamma_1}\pi_2)(f) = \bigl(\pi_1(f), P_{\gamma_1}\bigl(\pi_2(f)\bigr)\bigr), \quad \forall f \in \mathbb{R}^{\lambda^{\omega}}.$$

Since Z_{N-1} is isomorphic to $\mathbb{R}^{\lambda}$ and by (4) $X \oplus Y_0^{\lambda_1}$ contains no subspace isomorphic to $\mathbb{R}^{\lambda}$, it follows that $\pi_1 + P_{\gamma_1}\pi_2$ restricted to Y_{N-1} is not an isomorphism of Z_{N-1} into $X \oplus Y_0^{\lambda_1}$.

Hence there exists $f_2 \in Z_{N-1}$ such that $\|f_2\| = 1$, $\|\pi_1(f_2)\| \leqslant \epsilon/2^2$ and for every $\gamma \in [0, \lambda_1]$, it follows that $\|\pi_2(f_2)(\gamma)\| \leqslant \epsilon/2^2$.

Since $\pi_2(f_2) \in Y_0^{\lambda}$, pick $\lambda_2 \in [\lambda_1 + 1, \lambda)$ such that for every $\gamma \in [\lambda_2 + 1, \lambda]$ we have $\|\pi_2(f_2)(\gamma)\| \leqslant \epsilon/2^2$.

We may change f_2 to $-f_2$ and suppose that there exists $\xi_2 \in [0,\lambda)$ such that for every $\gamma \in (\lambda^{N}\xi_1 + \lambda^{N-1}\xi_2, \lambda^{N}\xi_1 + \lambda^{N-1}(\xi_2 + 1)]$, $f_2(\gamma) = 1$ holds.

Repeating this procedure N times we will find

- $f_1, f_2, \ldots, f_N \in \mathbb{R}^{\lambda^{\omega}}$,
- $\xi_1 < \xi_2 < \cdots < \xi_N < \lambda$,
- $\lambda_1 < \lambda_2 < \cdots < \lambda_N < \lambda$,

such that for every $1 \leqslant k \leqslant N$ and for every γ in

$$\bigl(\lambda^{N}\xi_1 + \lambda^{N-1}\xi_2 + \cdots + \lambda^{N-k+1}\xi_k, \lambda^{N}\xi_1 + \lambda^{N-1}\xi_2 + \cdots + \lambda^{N-k+1}(\xi_k + 1)\bigr]$$

we have:

- $f_k(\gamma) = \|f_k\| = 1$,
- $\operatorname{supp} f_2 \subset f_1^{-1}(1)$, $\operatorname{supp} f_3 \subset f_2^{-1}(1)$, $\ldots$, $\operatorname{supp} f_k \subset f_{k-1}^{-1}(1)$,

- $\|\pi_1(f_k)\| \leqslant \epsilon/2^k$,
- $\|\pi_2(f_k)(\gamma)\| \leqslant \epsilon/2^k$, $\forall\gamma \in [\lambda_k + 1, \lambda]$,
- $\|\pi_2(f_k)(\gamma)\| \leqslant \epsilon/2^k$, $\forall\gamma \in [0, \lambda_{k-1}]$, $k > 1$.

Let $f = f_1 + f_2 + \cdots + f_N$. Then it is obvious that $\|f\| = N$, $\|\pi_1(f)\| \leqslant \epsilon$, $\|\pi_2(f)\| \leqslant 1 + \epsilon$. Finally, by (5) we conclude that $aN \leqslant 1 + \epsilon$, which is absurd by the choice of N and ϵ. □

Before stating the next lemma, we recall some definitions from [9] and [12]. Let γ be an ordinal. A γ-sequence in a set A is a function $f : [1, \gamma[\to A$ and will be denoted by $(x_\theta)_{\theta<\gamma}$. If A is a topological space and β is an ordinal, we will say that the γ-sequence $(x_\theta)_{\theta<\gamma}$ is β-continuous if for every β-sequence of ordinals $(\theta_\xi)_{\xi<\beta}$ of $[0, \gamma]$ which converges to θ_β when ξ converges to β, we have that x_{θ_ξ} converges to x_{θ_β}.

Let X be a Banach space, α an ordinal number and φ a cardinal number. By X_α^φ we will denote the space of all $x^{**} \in X^{**}$ having the following property: for every ordinal $\beta < \alpha$ and φ-family $x^b = (x_\xi^*(b))_{\xi<\beta}$, $b \in \varphi$, of β-sequences of X^* such that there exists $M \in \mathbb{R}$ with $\|x_\xi^*(b)\| \leqslant M$ for every $b \in \varphi$ and $\xi < \beta$ and such that $x_\xi^*(b)(x) \xrightarrow{\xi\to\beta} 0$, $\forall x \in X$, uniformly in b, we have $x^{**}(x_\xi^*(b)) \xrightarrow{\xi\to\beta} 0$ uniformly in b.

Remark 2.2. Clearly X_α^φ is a closed subspace of X^{**} and $cX \subset X_\alpha^\varphi$, where cX is the canonical image of X in X^{**}. Observe also that if X has MP, then $X_\alpha^\varphi = cX$.

Let X be a Banach space and α an uncountable regular cardinal. Following [12, Definition 2.2], we set $[X]_\alpha = \bigcap_{\varphi<\alpha} X_\alpha^\varphi$. The following lemma is an extension of [10, Lemma 2.3].

Lemma 2.3. *Let X and Y be Banach spaces and α a regular ordinal. Then there exists an isomorphism $\Phi : X^{**} \oplus Y^{**} \to (X \oplus Y)^{**}$ satisfying*

(a) $\Phi(cX \oplus cY) = c(X \oplus Y)$.
(b) $\Phi([X]_\alpha \oplus [Y]_\alpha) = [X \oplus Y]_\alpha$.

Proof. Let $T : (X \oplus Y)^* \to X^* \oplus Y^*$ be the isomorphism given by $T(z^*) = (z^*_{|X}, z^*_{|Y})$, $\forall z \in (X \oplus Y)^*$. Then the isomorphism $T^* : (X^* \oplus Y^*)^* \to (X \oplus Y)^{**}$ is given by $(T^*z^{**})(w^*) = z^{**}(Tw^*)$, $\forall z^{**} \in (X^* \oplus Y^*)^*$ and $w^* \in (X \oplus Y)^*$.

Consider also the isomorphism $L : X^{**} \oplus Y^{**} \to (X^* \oplus Y^*)^*$ defined in the following way: $L(x^{**}, y^{**})(x^*, y^*) = x^{**}(x^*) + y^{**}(y^*)$, $\forall x^{**} \in X^{**}$, $\forall y^{**} \in Y^{**}$, $\forall x^* \in X^*$ and $\forall y^* \in Y^*$.

Put $\Phi = T^*L$. Then $\Phi(x^{**}, y^{**})(w^*) = x^{**}(w^*_{|X}) + y^{**}(w^*_{|Y})$, $\forall x^{**} \in X^{**}$, $\forall y^{**} \in Y^{**}$ and $w^* \in (X \oplus Y)^*$.

Now the proofs of the statements (a) and (b) are standard from the definition of the involved spaces. □

In preparation to the next results, as in [17], if α is an uncountable regular cardinal and γ is an arbitrary ordinal, by Λ_γ^α we will denote the set of all ordinals from $[0, \gamma]$ with cofinality at least α. Observe that $|\Lambda_{\alpha\xi}^\alpha| = |\xi|$, for every ordinal ξ.

Lemma 2.4. *Let α be an uncountable regular cardinal and γ an arbitrary ordinal. Suppose that X and Y are Banach spaces having MP. Then*

$$\frac{[X \oplus Y^\gamma]_\alpha}{c(X \oplus Y^\gamma)} \sim c_0\big(\Lambda_\gamma^\alpha, Y\big).$$

Proof. Let Φ be the function defined in Lemma 2.3. Thus

$$\frac{[X \oplus Y^\gamma]_\alpha}{c(X \oplus Y^\gamma)} = \frac{\Phi([X]_\alpha \oplus [Y^\gamma]_\alpha)}{\Phi(cX \oplus cY^\gamma)} \sim \frac{[X]_\alpha \oplus [Y^\gamma]_\alpha}{cX \oplus cY^\gamma}. \tag{6}$$

Since X has MP, it follows from the definition of $[X]_\alpha$ and Remark 2.2 that $[X]_\alpha = cX$. Therefore

$$\frac{[X]_\alpha \oplus [Y^\gamma]_\alpha}{cX \oplus cY^\gamma} = \frac{cX \oplus [Y^\gamma]_\alpha}{cX \oplus cY^\gamma} \sim \frac{[Y^\gamma]_\alpha}{cY^\gamma}. \tag{7}$$

On the other hand, Y also has MP. Consequently $[Y]_\alpha = cY$ and according to [12, Proposition 2.8], we have

$$\frac{[Y^\gamma]_\alpha}{cY^\gamma} \sim c_0\big(\Lambda_\gamma^\alpha, Y\big). \tag{8}$$

So by combining (6), (7) and (8) we get the desired result. □

Lemma 2.5. *Let X and Y be Banach spaces having MP and α an uncountable cardinal. Then for every $\gamma < \alpha$ we have*

$$\mathbb{R}^\alpha \not\hookrightarrow X \oplus Y^\gamma.$$

Proof. Suppose that $\mathbb{R}^\alpha \hookrightarrow X \oplus Y^\gamma$ for some $\gamma < \alpha$. We distinguish two cases:

Case 1. α is a regular cardinal. In this case, by [12, Lemma 2.4] we know that

$$\frac{[\mathbb{R}^\alpha]_\alpha}{c\mathbb{R}^\alpha} \hookrightarrow \frac{[X \oplus Y^\gamma]_\alpha}{c(X \oplus Y^\gamma)}. \tag{9}$$

On the one hand, [12, Proposition 2.8] implies that

$$\frac{[\mathbb{R}^\alpha]_\alpha}{c\mathbb{R}^\alpha} \sim \mathbb{R}. \tag{10}$$

On the other hand, by Lemma 2.4 we deduce that

$$\frac{[X \oplus Y^\gamma]_\alpha}{c(X \oplus Y^\gamma)} \sim c_0\big(\Lambda_\gamma^\alpha, Y\big) = \{0\}. \tag{11}$$

Thus putting (9), (10) and (11) together we reach a contradiction.

Case 2. α is a singular cardinal. Then, it is well known that there exists an uncountable regular cardinal $\gamma < \beta < \alpha$. Hence

$$\mathbb{R}^\beta \hookrightarrow \mathbb{R}^\alpha \hookrightarrow X \oplus Y^\gamma,$$

which leads to contradiction with Case 1. $\square$

We are ready to generalize [12, Lemma 2.9].

Lemma 2.6. *Let X and Y be Banach spaces having MP and α an uncountable regular cardinal. If $\mathbb{R}^{\alpha^2} \hookrightarrow X \oplus Y^\eta$ for some $\eta < \alpha^2$, then $c_0 \hookrightarrow Y$.*

Proof. Notice that $\mathbb{R}^\alpha \hookrightarrow \mathbb{R}^{\alpha^2}$. Then by the preceding lemma we cannot have $\eta < \alpha$. So $\alpha \leqslant \eta < \alpha^2$. Thus $\eta = \alpha\xi + \theta$ for some ordinals $\xi < \alpha$ and $\theta < \alpha$. Since that $\mathbb{R}^\eta \sim \mathbb{R}^{\alpha\xi}$ [17, Theorem 2], we have

$$Y^\eta \sim Y \hat{\hat{\otimes}} \mathbb{R}^\eta \sim Y \hat{\hat{\otimes}} \mathbb{R}^{\alpha\xi} \sim Y^{\alpha\xi}.$$

Therefore

$$\mathbb{R}^{\alpha^2} \hookrightarrow X \oplus Y^\eta \sim X \oplus Y^{\alpha\xi}. \tag{12}$$

Let I be a set with cardinality $|\xi|$. According to [12, Lemma 2.4] applied in (12),

$$\frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \hookrightarrow \frac{[X \oplus Y^{\alpha\xi}]_\alpha}{c(X \oplus Y^{\alpha\xi})}. \tag{13}$$

Next observe that by [12, Proposition 2.8],

$$c_0(\alpha) \sim \frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}}. \tag{14}$$

Moreover, Lemma 2.4 implies that

$$\frac{[X \oplus Y^{\alpha\xi}]_\alpha}{c(X \oplus Y^{\alpha\xi})} \sim c_0(I, Y). \tag{15}$$

Hence by (13), (14) and (15) we see that

$$c_0(\alpha) \hookrightarrow c_0(I, Y).$$

Since $|I| < \alpha$, by [11, Lemma 2.4] we infer that $c_0 \hookrightarrow X$. $\square$

We end this section by improving [12, Lemma 2.10].

Lemma 2.7. *Let $\omega < \alpha \leqslant \xi \leqslant \eta$ be such that $|\eta| = \alpha$. Put $\alpha_0 = \alpha$ if α is a singular cardinal and $\alpha_0 = \alpha^2$ if α is a regular cardinal. Suppose that X and Y are Banach spaces having MP such that Y contains no subspace isomorphic to c_0. If $\mathbb{R}^\eta \hookrightarrow X \oplus Y^\xi$ with $\alpha_0 \leqslant \xi$, then $\mathbb{R}^\eta \hookrightarrow \mathbb{R}^\xi$.*

Proof. We introduce two sets of ordinals

$$I_1 = \{\theta\colon\ \bar{\theta} = \bar{\alpha},\ \alpha_0 \leqslant \theta,\ \mathbb{R}^\theta \not\hookrightarrow \mathbb{R}^\gamma,\ \forall \gamma < \theta\},$$
$$I_2 = \{\theta\colon\ \bar{\theta} = \bar{\alpha},\ \alpha_0 \leqslant \theta,\ \mathbb{R}^\theta \not\hookrightarrow X \oplus Y^\gamma,\ \forall \gamma < \theta\}.$$

First of all we will prove that $I_1 = I_2$. Clearly $I_2 \subset I_1$. Observe that by Lemmas 2.5 and 2.6 we deduce that $\alpha_0 \in I_2$. Now, assume that I_2 is a proper subset of I_1. Let α_1 be the least element of $I_1 \backslash I_2$. We have $\alpha_0 < \alpha_1$. Since $\alpha_1 \notin I_2$, there exists an ordinal $\gamma_1 < \alpha_1$ such that $\mathbb{R}^{\alpha_1} \hookrightarrow X \oplus Y^{\gamma_1}$.

Let $\alpha_2 = \min\{\gamma,\ \alpha_0 \leqslant \gamma < \alpha_1\colon\ \mathbb{R}^{\alpha_1} \hookrightarrow X \oplus Y^\gamma\}$. We have $\alpha_2 \leqslant \gamma_1$. Now, we will show that $\alpha_2 \in I_1$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $\mathbb{R}^{\alpha_2} \hookrightarrow \mathbb{R}^{\gamma_2}$. Therefore $X_2^\alpha \hookrightarrow X_2^\gamma$. Consequently $\mathbb{R}^{\alpha_1} \hookrightarrow X \oplus Y^{\gamma_2}$, in contradiction with the definition of α_2.

So $\alpha_2 \in I_1$ and since $\alpha_2 < \alpha_1$, it follows from the definition of α_1 that $\alpha_2 \in I_2$. That is, $\mathbb{R}^{\alpha_2} \not\hookrightarrow X \oplus Y^\gamma$, $\forall \gamma < \alpha_2$. Thus by Lemma 2.1, we conclude that $\mathbb{R}^{\alpha_2^\omega} \not\hookrightarrow X \oplus Y^{\alpha_2}$.

On the other hand, note that if $\alpha_1 < {\alpha_2}^\omega$, then by [17, Theorem 1] and [17, Theorem 2], $\mathbb{R}^{\alpha_1} \sim \mathbb{R}^{\alpha_2}$, which is absurd by the definition of α_1. Consequently ${\alpha_2}^\omega \leqslant \alpha_1$ and $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow \mathbb{R}^{\alpha_1}$. Furthermore, by the definition of α_2, $\mathbb{R}^{\alpha_1} \hookrightarrow X \oplus Y^{\alpha_2}$. Therefore $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow X \oplus Y^{\alpha_2}$, in contradiction with what we have just proved above. Hence $I_1 = I_2$.

Next, to complete the proof of the lemma, suppose that $\mathbb{R}^\eta \not\hookrightarrow \mathbb{R}^\xi$ and let $\xi_1 = \min\{\theta\colon\ \mathbb{R}^\eta \hookrightarrow \mathbb{R}^\theta\}$. Hence $\xi < \xi_1 \leqslant \eta$ and $\mathbb{R}^{\xi_1} \not\hookrightarrow \mathbb{R}^\gamma$, $\forall \gamma < \xi_1$. In particular, $\xi_1 \in I_1 = I_2$, which is absurd, because $\mathbb{R}^{\xi_1} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X \oplus Y^\xi$. $\square$

3. Proof of Theorem 1.1

Initially we will prove that statement (b) implies statement (a). First suppose that $\mathbb{R}^\xi \sim \mathbb{R}^\eta$. Then

$$Y^\xi \sim Y \hat{\hat{\otimes}} \mathbb{R}^\xi \sim Y \hat{\hat{\otimes}} \mathbb{R}^\eta \sim Y^\eta.$$

Consequently

$$X \oplus Y^\xi \sim X \oplus Y^\eta.$$

Next assume that $\mathbb{R}^\xi \sim \mathbb{R}^{\alpha p}$, $\mathbb{R}^\eta \sim \mathbb{R}^{\alpha q}$ and $Y^p \sim Y^q$, for some uncountable regular cardinal α and finite ordinals p and q. Thus

$$Y^{\alpha p} \sim (Y^p)^\alpha \sim (Y^q)^\alpha \sim Y^{\alpha q}.$$

Hence

$$Y^\xi \sim Y \hat{\hat{\otimes}} \mathbb{R}^\xi \sim Y \hat{\hat{\otimes}} \mathbb{R}^{\alpha p} \sim Y^{\alpha p} \sim Y^{\alpha q} \sim Y \hat{\hat{\otimes}} \mathbb{R}^{\alpha q} \sim Y \hat{\hat{\otimes}} \mathbb{R}^\eta \sim Y^\eta.$$

Therefore again we conclude that

$$X \oplus Y^\xi \sim X \oplus Y^\eta.$$

We pass now to prove that statement (a) implies statement (b). Assume then that

$$X \oplus Y^\xi \sim X \oplus Y^\eta. \tag{16}$$

First of all we will show that $|\xi| = |\eta|$. Assume that $|\xi| \neq |\eta|$. We may suppose without loss of generality that $|\eta| > |\xi| = \alpha$. We distinguish two cases:

Case 1. α is a singular cardinal or α is a regular cardinal with $\alpha^2 \leqslant \xi$. In this case, since $\xi^\omega < \eta$ by (16) we have

$$\mathbb{R}^{\xi^\omega} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X \oplus Y^\eta \sim X \oplus Y^\xi. \tag{17}$$

So by Lemma 2.7, $\mathbb{R}^{\xi^\omega} \hookrightarrow \mathbb{R}^\xi$, which is a contradiction with [2, Lemma 2].

Case 2. α is a regular cardinal with $\xi < \alpha^2$. Thus write $\xi = \alpha\xi' + \gamma$ with $\xi', \gamma < \alpha$. By [17, Theorem 2], $\mathbb{R}^\xi \sim \mathbb{R}^{\alpha\xi'}$ and hence $Y^\xi \sim Y^{\alpha\xi'}$. Since $\alpha^2 < \eta$, by (16) we deduce

$$\mathbb{R}^{\alpha^2} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X \oplus Y^\eta \sim X \oplus Y^{\alpha\xi'}. \tag{18}$$

Applying [12, Lemma 2.4] in (18) we see that

$$\frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \hookrightarrow \frac{[X \oplus Y^{\alpha\xi'}]_\alpha}{c(X \oplus Y^{\alpha\xi'})}. \tag{19}$$

Now, let I be a set with cardinality $|\xi'|$. Then according to [12, Proposition 2.8] and Lemma 2.4 we obtain respectively

$$c_0(\alpha) \sim \frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \quad \text{and} \quad \frac{[X \oplus Y^{\alpha\xi'}]_\alpha}{c(X \oplus Y^{\alpha\xi'})} \sim c_0(I, Y). \tag{20}$$

Hence by (19) and (20) we obtain

$$c_0(\alpha) \hookrightarrow c_0(I, Y).$$

Since $|I| < \alpha$, it follows from [11, Lemma 2.4] that $c_0 \hookrightarrow Y$. This contradiction finishes the proof of $|\xi| = |\eta|$.

Next assume that (16) holds for some ordinals $\xi \leqslant \eta$ with $|\xi| = |\eta| = \alpha$. It is convenient to consider two cases:

Case 1. α is a singular cardinal or α is a regular cardinal with $\alpha^2 \leqslant \xi$.

Notice that if $\xi^\omega \leqslant \eta$, then (17) also holds. So, as we have just show above, by Lemma 2.7 and [2, Lemma 2], we obtain a contradiction. Thus $\eta < \xi^\omega$ and by [17, Theorem 1], we conclude that $\mathbb{R}^\xi \sim \mathbb{R}^\eta$.

Case 2. α is a regular cardinal with $\xi < \alpha^2$. Write $\xi = \alpha\xi' + \gamma$ with $\xi', \gamma < \alpha$. First we will prove that $\eta \leqslant \alpha^2$. Indeed, suppose that $\alpha^2 < \eta$. Then, by (16) we have

$$\mathbb{R}^{\alpha^2} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X \oplus Y^\eta \sim X \oplus Y^\xi. \tag{21}$$

Thus applying [12, Lemma 2.4] in (21) we get

$$\frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \hookrightarrow \frac{[X \oplus Y^\xi]_\alpha}{c(X \oplus Y^\xi)}. \tag{22}$$

Moreover, according to [12, Proposition 2.8] and Lemma 2.4 we infer respectively

$$c_0(\alpha) \sim \frac{[\mathbb{R}^{\alpha^2}]_\alpha}{c\mathbb{R}^{\alpha^2}} \quad \text{and} \quad \frac{[X \oplus Y^\xi]_\alpha}{c(X \oplus Y^\xi)} \sim c_0\big(\Lambda_\xi^\alpha, Y\big). \tag{23}$$

Therefore by (22) and (23) we deduce

$$c_0(\alpha) \hookrightarrow c_0\big(\Lambda_\xi^\alpha, Y\big).$$

Since $|\Lambda_\xi^\alpha| = |\xi'| < \alpha$, [11, Lemma 2.4] implies that $c_0 \hookrightarrow Y$, a contradiction.

Thus we can assume that $\eta \leqslant \alpha^2$. Since (16) holds, it follows from [12, Remark 2.3] that

$$\frac{[X \oplus Y^\xi]_\alpha}{c(X \oplus Y^\xi)} \sim \frac{[X \oplus Y^\eta]_\alpha}{c(X \oplus Y^\eta)}. \tag{24}$$

Write $\eta = \alpha\eta' + \delta$, with $\eta', \delta \leqslant \alpha$. Then by (24) and Lemma 2.4 we obtain

$$c_0\big(\Lambda_\xi^\alpha, Y\big) \sim c_0\big(\Lambda_\eta^\alpha, Y\big). \tag{25}$$

Since $|\Lambda_\xi^\alpha| = |\xi'|$, $|\Lambda_\eta^\alpha| = |\eta'|$ and $c_0 \not\hookrightarrow Y$, it follows from [11, Lemma 2.4] that $|\xi'|$ is infinite if and only if $|\eta'|$ is infinite and in the affirmative case $|\xi'| = |\eta'|$. Thus by [17, Theorem 2], $\mathbb{R}^\xi \sim \mathbb{R}^\eta$.

Finally, in the case where ξ' and η' are finite, observe that if $|\xi'| = |\eta'|$, then by [17, Theorem 2] $\mathbb{R}^\xi \sim \mathbb{R}^\eta$. Otherwise $|\xi'| \neq |\eta'|$ and by [17, Theorem 2], $\mathbb{R}^\xi \sim \mathbb{R}^{\alpha\xi'}$, $\mathbb{R}^\eta \sim \mathbb{R}^{\alpha\eta'}$. Further, according to (25) we have $Y^{\xi'} \sim Y^{\eta'}$.

Thus the theorem is proved.

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